$\| C_{(k)} \| = k$ and $g/e < 1$, it follows that

$$k \sum_{\alpha_1+\alpha_2=k} |u_{\alpha_1\alpha_2}| \leqslant \frac{1}{e-g} \sum_{\alpha_1+\alpha_2=k} |v_{\alpha_1\alpha_2}|$$

whence we obtain $\| u \|_1 \leqslant c \| v \|_2$, $c = 1/(e+g)$.

The proof for the other cases is simpler.

Consequently, the operator $F_s'(S_2, 0)$ indeed has a bounded inverse.

By the Implicit Functions Theorem /4/, for small ε a unique solution $S(x, \varepsilon)$ of Eq.(6) exists, differing only slightly from $S_2(x)$. This proves the existence of an analytic solution S^+ of Eq.(1). Since the assumptions of the theorem are invariant when T_1 is replaced by $-T_1$ in the formula for the Lagrangian, this implies the existence of a second solution S.

REFERENCES

1. MALKIN I.G., Theory of the Stability of Motion, Nauka, Moscow, 1966.
2. BULATOVICH R.M., The existence of solutions of the Hamilton-Jacobi equation in the neighbourhood of non-degenerate equilibrium positions. Prikl. Mat. Mekh., 47, 2, 1983.
3. BOLOTIN S.V. and KOZLOV V.V., On asymptotic solutions of the equations of dynamics. Vestnik Leningrad. Gos. Univ., Ser. 1, Mat. Mekh., 4, 1980.
4. KOLMOGOROV A.N. and FOMIN S.V., Elements of the Theory of Functions and Functional Analysis, Nauka, Moscow, 1981.

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A REPRESENTATION OF THE SOLUTIONS OF THE GENERALIZED CAUCHY-RIEMANN EQUATIONS AND ITS APPLICATIONS*

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A method of integral representations for the generalized Cauchy-Riemann system in terms of an arbitrary analytic function, similar to the well-known Whittaker-Polozhii representation /1/, is developed. The representation includes various well-known results as special cases, and the limiting case leads to the classical representation of the theory of a generalized axisymmetric potential. The representations established are used to reduce mixed problems for the system to paired equations and then to a Fredholm equation of the second kind. At the same time, a device is described for regularizing paired equations, and a case in which a closed solution exists is presented.

The results are extended to a sytem of more-general form and also to second-order equations, whose type and dimensionality are not essential. It is shown that the integral operators constructed here convert the solution of a parabolic or hyperbolic equation with variable coefficients into a solution of the classical equations of heat conduction and wave propagation, thus furnishing an explicit representation for solutions of the corresponding Cauchy problems.

The effectiveness of the approach is demonstrated with reference to the problem of inflow in a fissure in an inhomogeneous layer of finite

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width. Simple formulae for the pressure and discharge of fissures are presented.

Special cases of the system

$$\frac{\partial H}{\partial \theta} = -A(m \operatorname{cth} m\sigma)^{1-2\mu}\frac{\partial \psi}{\partial \sigma}, \quad \frac{\partial H}{\partial \sigma} = A(m \operatorname{cth} m\sigma)^{1-2\mu}\frac{\partial \varphi}{\partial \sigma} \tag{0.1}$$

where A, μ, m^2 are arbitrary real parameters, have found wide application in problems of the mechanics of continuous media and physics in general.

Thus, if $2\mu = -1$ we obtain the well-known approximation /2/ of the Chaplygin equations, which has been used for the effective solution of problems in subsonic $(A > 0)$ and supersonic gas- and wave-dynamics $(A < 0)$.

If $2\mu = p$, $mp = 1$, we have a system of equations for non-linear filtration on the hodograph plane /3/ for the law of resistance /3-5/

$$\Phi(w) = (w^{2/p} \pm \lambda^{2/p})^{p/2}, \quad w = \lambda \begin{Bmatrix} \operatorname{sh}^p \sigma/p \\ \operatorname{ch}^p \sigma/p \end{Bmatrix} \tag{0.2}$$

which describes various types of rheological behaviour, from thixotropic to pseudoplastic, and contains practically all the most commonly used models, with the exception of the power model. In particular, putting $p = 2\,(\mu = -1, 2m = 1)$ we obtain the system of equations of filtration problems with limit gradient, studied in detail in /3/. It is also known /6-8/ that for $2\mu = -1 - 2k$ (where k is an integer) the solution of the system can be expressed in terms of a solution of the Cauchy-Riemann system in the form of a finite differential operator of order $k + 1$. This considerably simplifies the determination of exact solutions of mixed problems for the initial system in non-canonical regions (the case $k = 0$ corresponds to the model of a filtration law with limit gradient /9/ and its one-parametric generalization /10/).

Finally, as $m \to 0$ we obtain the system of equations of an axisymmetric potential, whose solution admits of an integral representation in terms of an arbitrary analytic function /1/; this provides the basis for a universal method of reducing mixed problems for the above system to Fredholm equations of the second kind /1, 11-14/.

1. Eliminating the function $\psi(\sigma, \theta)$ from the initial system (0.1), we obtain the second-order equation

$$\frac{\partial^2 H}{\partial \sigma^2} + \frac{2m(1-2\mu)}{\operatorname{sh} 2m\sigma}\frac{\partial H}{\partial \sigma} + \frac{\partial^2 H}{\partial \theta^2} = 0 \tag{1.1}$$

Looking for a solution of this equation in the form

$$H(\sigma, \theta) = \sum_{n=1}^{\infty} A_n Z_n(t) \exp n\theta, \quad t = -\operatorname{sh}^2 m\sigma \tag{1.2}$$

we see that $Z_n(t)$ is a hypergeometric function

$$Z_n(t) = {}_2F_1(\alpha, \beta; \gamma; t), \quad -\alpha = \beta = n/2m, \quad \gamma = 1 - \mu \tag{1.3}$$

Since the condition for the convergence of the integral in Euler's formula /15/

$$\mathrm{B}(\gamma, \gamma - \beta)\,{}_2F_1(\alpha, \beta; \gamma; t) = \int_0^1 \tau^{\beta-1}(1-\tau)^{\gamma-\beta-1}(1-\tau t)^{-\alpha}\, d\tau$$

$(\gamma > \beta)$ need not be satisfied in this case, we shall use the identities /15/

$$2\,{}_2F_1(-\alpha, \alpha; {}^1/_2; t) = (\sqrt{1-t} + i\sqrt{t})^{2\alpha} + (\sqrt{1-t} - i\sqrt{t})^{2\alpha} \tag{1.4}$$

$$\mathrm{B}(\gamma, \varepsilon)\,{}_2F_1(-\alpha, \alpha; \gamma + \varepsilon; t) = \int_0^1 y^{\gamma-1}(1-y)^{\varepsilon-1}\,{}_2F_1(-\alpha, \alpha; \gamma; ty)\, dy$$

(where $(B(\gamma, \varepsilon)$ is the beta function), putting $2\gamma = 1$, $2\varepsilon = 1 - 2\mu$ in the second identity. We obtain

$$\mathrm{B}({}^1/_2, {}^1/_2 - \mu)\,{}_2F_1(-\alpha, \alpha, 1-\mu; t) = \int_0^1 {}_2F_1(-\alpha, \alpha, {}^1/_2; ty)\,\frac{(1-y)^{-1/2-\mu}}{\sqrt{y}}\, dy \tag{1.5}$$

We now substitute (1.5) into (1.2) and interchange the summation and integral symbols, assuming that this is legitimate. The result is

$$H(\sigma,\theta) = {}^1/_2 \int_0^1 \sum_{n=1}^{\infty} S_n(\theta, ty) \frac{(1-y)^{-1/_2-\mu}}{\sqrt{y}}\, dy$$

$$S_n(\theta, ty) = 2B_n\, {}_2F_1(-\alpha, \alpha; {}^1/_2; ty), \quad A_n = \mathrm{B}({}^1/_2, {}^1/_2 - \mu) B_n$$

Using Eqs.(1.3) and (1.4), we transform the sum in the integrand as follows:

$$\sum_{n=1}^{\infty} S_n(\theta, ty) = \sum_{n=1}^{\infty} B_n (z_+^n + z_-^n)$$

$$z_\pm = \exp s_\pm, \quad s_\pm = \theta \pm i m^{-1} \operatorname{Arsh} \sqrt{-ty}$$

Suppose now that B_n are the coefficients of the Taylor series of an analytic function $q(z)$. Then we have

$$\sum_{n=1}^{\infty} B_n z_\pm^n = q(z_\pm) = g(s_\pm)$$

Transforming to variables $u = \sqrt{y}$ and σ ($t = -\mathrm{sh}^2\, m\sigma$), we conclude that

$$H(\sigma, \theta) = \int_0^1 (1-u^2)^{-\nu} [g(s_+) + g(s_-)]\, du \tag{1.6}$$

$$s_\pm = \theta \pm i\xi, \quad \xi = m^{-1} \operatorname{Arsh} u \,\mathrm{sh}\, m\sigma$$

The function $\psi(\sigma, \theta)$ can be determined in a similar fashion, but it is easier to use the following property of system (0.1).

Define

$$\psi = \left(\frac{\mathrm{th}\, m\sigma}{m}\right)^{1-2\mu} \frac{\partial V}{\partial \sigma}, \quad H = \frac{\partial V}{\partial \theta}$$

and insert these relations into the initial system. It then transpires that $V(\sigma, \theta)$ satisfies exactly the same equation as $H(\sigma, \theta)$. Thus

$$H(\rho, \theta) = \int_0^1 [G(\theta + i\xi) + G(\theta - i\xi)] (1-u^2)^{-\nu}\, du \tag{1.7}$$

$$\psi = i \left(\frac{\mathrm{th}\, m\sigma}{m}\right)^{1-2\mu} \int_0^1 [G(\theta + i\xi) - G(\theta - i\xi)] \frac{(1-u^2)^{-\nu} u \,\mathrm{ch}\, m\sigma}{\sqrt{1 + u^2 \mathrm{sh}^2 m\sigma}}\, du$$

$$G(s) = g'(s)$$

We have thus established the desired integral representation of a solution of system (0.1) in terms of an arbitrary analytic function.

Applied to problems of filtration with a limit gradient /3/, formula (1.7) immediately yields the asymptotic behaviour of $\psi(\sigma, \theta)$ near the boundary of the stagnation zone, where $\sigma = 0$:

$$\psi(0, \theta) \approx a(\theta) \sigma^{2-2\mu} \approx a_1(\theta) w^{1-\frac{1}{\mu}}$$

These representations are also convenient when the distribution of the discharge $H(\sigma, \theta)$ is known along the boundary of the stagnation zone, $H(0, \theta) = f(\theta)$. Setting $\sigma = 0$ in the expression for $H(\sigma, \theta)$, we at once determine the arbitrary function $G(s)$:

$$H(0, \theta) = f(\theta) = 2G(\theta) \int_0^1 (1-u^2)^{-\nu}\, du$$

In view of this state of affairs, we can also consider inverse problems of lengthwise displacement of rigid-plastic bodies, by virtue of the analogy /16/ with filtration problems with limit gradient. Indeed, if we assume that the borderline between the plastic and the rigid zones (the rigid zone corresponds to the stagnation zone in filtration problems) has a given curvature $\varkappa(\theta)$, it follows from the formula of transition to the physical plane /3/ and from (1.7) that

$$\varkappa(\theta) = 2G'(\theta)\int_0^0 (1-u^2)^{-\nu}du$$

Transforming to the new variable ξ (see the last equality of (1.6)) in (1.7), we define

$$2G(\theta + i\xi) = U(\xi, \theta) - iV(\xi, \theta)$$

The result is the following representation for a solution of system (0.1):

$$H(\sigma, \theta) = \operatorname{sh}^{2\mu} m\sigma \int_0^\sigma U(\xi, \theta)\frac{m \operatorname{ch} m\xi}{\Delta^\nu}\, d\xi \tag{1.8}$$

$$\psi(\sigma, \theta) = (m \operatorname{ch} m\sigma)^{2\mu}\int_0^\sigma V(\xi, \theta)\frac{\operatorname{sh} m\xi}{\Delta^\nu}\, d\xi$$

$$\Delta = \operatorname{sh}^2 m\sigma - \operatorname{sh}^2 m\xi$$

where $U(\xi, \theta)$ and $V(\xi, \theta)$ $(V(0, \theta) = 0)$ are arbitrary harmonic functions; this generalizes the known representations of solutions of the equations of a generalized axisymmetric potential /1/ and indeed yields the latter when $m \to 0$. The fact that $H(\sigma, \theta)$ and $\psi(\sigma, \theta)$ in (1.8) indeed satisfy the initial system can be verified directly.

We shall now exhibit a solution of system (0.1) that differs from (1.8) in that the integration is performed from σ to ∞, and $U(\xi, \theta)$ and $V(\xi, \theta)$ are replaced by certain other harmonic functions $U_1(\xi, \theta)$ and $V_1(\xi, \theta)$. Concerning the latter we require that

$$F(\zeta) = U_1 - iV_1 \sim \exp(2\mu - \xi)\zeta, \quad \zeta = \xi + i\theta, \ \varepsilon > 0$$

as $\zeta \to \infty$ uniformly in θ (this guarantees convergence of the integrals) and establish their relation with $U(\xi, \theta)$ and $V(\xi, \theta)$.

If one tries to do this using formulae for the analytic continuation of the hypergeometric function in (1.5) (see /15/), the results involve quite cumbersome manipulations. Much to be preferred is another approach, based on a generalization of an identity in /17/, p.574, which links Cauchy integrals with integrals of fractional order:

$$\int_0^x \left[\frac{\sin \nu\pi}{\pi}\int_0^\infty \frac{\varphi(\tau, \theta)}{\tau - t}\left(\frac{\tau}{t}\right)^{1-\nu} d\tau - \cos \nu\pi\varphi(t, \theta)\right]\frac{dt}{(x-t)^\nu} = \int_x^\infty \frac{\varphi(t, \theta)}{(t-x)^\nu}\, dt \tag{1.9}$$

In the representation for $H(\sigma, \theta)$ we now transform to variables $x = \operatorname{sh}^2 m\sigma$, $t = \operatorname{sh}^2 m\xi$, to obtain

$$U(x, \theta) = x^{\nu - 1/2}\int_0^x \frac{U(t, \theta)}{2\sqrt{t}(x-t)^\nu}\, dt \tag{1.10}$$

Now, putting

$$U(t, \theta) = 2\sqrt{t}\left[\lambda\int_0^\infty \frac{\varphi(\tau, \theta)}{\tau - t}\left(\frac{\tau}{t}\right)^{1-\nu} d\tau - \varphi(t, \theta)\right], \quad \lambda = \frac{1}{\pi}\operatorname{tg}\nu\pi \tag{1.11}$$

in (1.10) and using (1.9), we obtain the required representation:

$$H(x, \theta) = x^{\nu - 1/2}\int_x^\infty \frac{\varphi(t, \theta)}{(t-x)^\nu}\, dt \tag{1.12}$$

in which we must still express $\varphi(t, \theta)$ in terms of $U(t, \theta)$. But Eq.(1.11) is a singular integral equation for $\varphi(t, \theta)$, which reduces to an equation with constant coefficients:

$$-\varphi_1(t, \theta)\cos \nu\pi + \frac{\sin \nu\pi}{\pi}\int_0^\infty \frac{\varphi_1(\tau, \theta)}{\tau - t}\, d\tau = \varphi_2(t, \theta) \tag{1.13}$$

$$\varphi_1(t, \theta) = \varphi(t, \theta)t^{1+\nu}, \quad \varphi_2(t, \theta) = {}^1/_2 t^{1/2 - \nu}\, U(t, \theta)$$

Taking the Mellin transform /18/ of this equation and simplifying, we obtain

$$\varphi_1(s, \theta) = -\varphi_2(s, \theta)\cos \nu\pi - \varphi_2(s, \theta)\sin \nu\pi \operatorname{ctg}(s + 1 - \nu)\pi$$

$$\varphi_i(s, \theta) = \int_0^\infty \varphi_i(t, \theta)\, t^{s-1}\, dt$$

Using the Convolution Theorem and returning to the original functions, in view of (1.13), we obtain

$$\varphi(t, \theta) = \frac{U(t, \theta)}{2\sqrt{t}} \cos \nu\pi - \frac{\sin \nu\pi}{2\pi} \int_0^\infty \frac{U(\tau, \theta)}{\sqrt{\tau}(\tau - t)}\, d\tau \tag{1.14}$$

Inserting (1.14) into (1.10) and transforming to variables σ, ξ, noting moreover that $U(\sigma, \theta)$ is an even function of σ, we obtain the desired representation:

$$H(\sigma, \theta) = \operatorname{Re} \operatorname{sh}^{2\mu} m\sigma \int_\sigma^\infty M(W) \frac{m \operatorname{ch} m\xi}{\Delta_1^{\nu}}\, d\xi \tag{1.15}$$

$$\psi(\sigma, \theta) = \operatorname{Im}(m \operatorname{ch} m\sigma)^{2\mu} \int_\sigma^\infty M(W) \frac{\operatorname{sh} m\xi}{\Delta_1^{\nu}}\, d\xi$$

$$M(W) = \sin \nu\pi W(\xi, \theta) - \frac{\cos \nu\pi}{\pi} \int_{-\infty}^\infty \frac{W(t, \theta)\, m}{\operatorname{sh} m(t - \xi)}\, dt$$

$$W = U(\xi, \theta) + iV(\xi, \theta), \quad \Delta_1 = (\operatorname{sh}^2 m\xi - \operatorname{sh}^2 m\sigma)$$

(In order to determine the formula for $\psi(\sigma, \theta)$, just replace $V(t, \theta)$ in the previous arguments by $U(t, \theta)\sqrt{1 + t^2}$ and proceed as before).

2. Before finding formulae for inverting the integral operators (1.8) and (1.15), we offer some preliminary remarks. If $2\mu = 1 - 2k$, where k is an integer, the solution of the integral equations obtained from (1.8) for $U(\xi, \theta)$ and $V(\xi, \theta)$ is sought by $(k + 1)$-fold differentiation, which yields previously known results /6-8/.

According to /7, 8/, the solutions of system (0.1) and the system obtained from it by replacing μ with $\mu + 1$ satisfy the following relations:

$$\psi_\mu = \psi_{\mu+1} - \frac{\operatorname{th} m\sigma}{2\mu m} \frac{\partial \psi_{\mu+1}}{\partial \sigma} \tag{2.1}$$

$$H_\mu = m^2 H_{\mu+1} - \frac{m \operatorname{cth} m\sigma}{2\mu} \frac{\partial H_{\mu+1}}{\partial \sigma}$$

By virtue of this property, it will suffice to consider the case $|2\mu| < 1$ (for $2\mu = \pm 1$ the systems degenerate into Cauchy-Riemann systems), but in that case (1.8) and (1.15) are generalizations of Abel's equations /1, 11-14/, for whose solution standard formulae are available (see, e.g., /14/, p.71). We thus obtain the following pair of inversion formulae for (1.8):

$$U(\sigma, \theta) = \frac{2}{\pi} \frac{\cos \mu\pi}{\operatorname{ch} m\sigma} \frac{\partial}{\partial \sigma} \int_0^\sigma \frac{(\operatorname{sh} m\xi)^{1-2\mu} \operatorname{ch} m\xi}{\Delta^{1-\nu}} H(\xi, \theta)\, d\xi \tag{2.2}$$

$$V(\sigma, \theta) = \frac{2}{\pi} \frac{\cos \mu\pi}{\operatorname{ch} m\sigma} \frac{\partial}{\partial \sigma} \int_0^\sigma \frac{(m \operatorname{ch} m\xi)^{1-2\mu} \operatorname{sh} m\xi}{\Delta^{1-\nu}} \psi(\xi, \theta)\, d\xi$$

To derive inversion formulae for (1.15), we need only replace the upper and lower limits of integration in (2.2) by ∞ and σ, respectively, and replace Δ by Δ_1.

The availability of transformation formulae (1.8) and (1.15), together with their inversion formulae (2.2), makes it possible to reduce mixed boundary-value problems for system (0.1) in regions such as a half-strip to problems in the theory of analytic functions; the procedure is analogous to that used in the theory of an axisymmetric potential /1, 11-14/.

We shall assume without loss of generality that a homogeneous Dirichlet or Neumann condition is stipulated on the boundary $\theta = \theta_0$; at $\theta = 0$ the homogeneous Neumann condition is given on the part of the boundary with $\sigma > \sigma_0 = a$ - a typical situation in mixed problems (if the boundary conditions at the boundaries of the half-strip are not mixed, application of

(2.2) immediately yields a Dirichlet or Neumann problem for the Laplace equation, and a closed solution can be constructed by successive quadratures).

As to the mixed problem

$$H(\sigma, \theta_0) = 0, \quad 0 < \sigma < \infty$$
$$H(\sigma, 0) = F(\sigma), \ 0 < \sigma < a, \ H(\sigma, 0) = 0, \ \sigma > a$$

application of (1.18), (1.15) and (2.2) gives the following boundary-value problem for an analytic function $F(\zeta) = U + iV$:

$$U(\sigma, \theta_0) = 0, \ |\sigma| < \infty, \ U(\sigma, 0) = F_1(\sigma), \ |\sigma| < a \tag{2.3}$$
$$\sin \mu\pi \frac{\partial U}{\partial \theta}(\sigma, 0) - \frac{\cos \mu\pi}{\pi} \int_{-\infty}^{\infty} \frac{\partial U}{\partial \theta}(t, 0) \frac{m\, dt}{\operatorname{sh} m(t-\sigma)} = 0, \quad |\sigma| > a$$

where the boundary conditions for $U(\sigma, \theta)$ - an even function - are extended symmetrically to negative σ, and $F_1(\sigma)$ is defined by the right-hand side of the first formulae in (2.2).

Using Fourier integrals /17, 18/, one can show that the paired equations of the boundary-value problem (2.3) reduce to a complete singular integral equation /17/

$$\varphi(\sigma) + \int_a^{\infty} \varphi(t) K(\sigma, t)\, dt = \delta R(\sigma) \tag{2.4}$$
$$K(\sigma, t) = \lambda \frac{\operatorname{sh} \varepsilon t \operatorname{ch} \varepsilon\sigma}{\operatorname{sh}^2 \varepsilon t - \operatorname{sh}^2 \varepsilon\sigma} + \int_0^{\infty} \frac{\operatorname{sh} k(\varepsilon - \theta_0) \cos kt \cos k\sigma}{\operatorname{sh} k\theta_0 \operatorname{ch} k\varepsilon}\, dk$$
$$R(\sigma) = \int_0^a F_1(t) \cos t\sigma\, dt, \quad \lambda = \frac{1}{\pi} \operatorname{tg} \mu\pi, \quad \lambda\delta = \sin \mu\pi, \quad \varepsilon = \frac{\pi}{2m}$$

which can be regularized by well-known methods /17/.

Special consideration should be given to the case $2m\theta_0 = \pi$. We will first present a formula for solving the following boundary-value problem for an analytic function:

$$U(\xi, \theta_0) = 0, \ V(\xi, 0) = V(\xi), \ |\xi| < \infty$$

The formula, derived in the same way as Schwarz's formula for poles /18/, is

$$\frac{d}{d\zeta} U + iV = \frac{1}{2\theta_0} \int_{-\infty}^{\infty} \frac{V'(t)}{\operatorname{sh} \alpha(t-\zeta)}\, dt, \quad \alpha = \frac{\pi}{2\theta_0} \tag{2.5}$$

Using the Cauchy-Riemann conditions, we rewrite problem (2.3) as

$$\sin \mu\pi \frac{\partial V}{\partial \xi}(\xi, 0) - \frac{\cos \mu\pi}{\pi} \int_{-\infty}^{\infty} \frac{\partial V}{\partial t}(t, 0) \frac{m}{\operatorname{sh} m(t-\zeta)}\, d\xi = 0$$

and, using (2.5), replace the integral appearing here as follows:

$$\frac{\partial V}{\partial \xi}(\xi, 0) \sin \mu\pi - \frac{\partial U}{\partial \xi}(\xi, 0) \cos \mu\pi = \operatorname{Re} \frac{diV}{d\zeta} e^{-i\pi\mu} = 0$$

Thus, mixed problems for system (0.1) in a strip of width $\theta_0 = \pi/2m$ (in the integral Eq.(2.4) the regular part of the kernel vanishes) can be solved in closed form, as in the analogous problem for the equations of a generalized axisymmetric potential in a half-space /1, 11-14/. Such problems clearly reduce to a Riemann problem with discontinuous coefficients /17/ in the strip

$$\operatorname{Re} W = 0, \quad |\sigma| < \infty, \ \theta = \theta_0 \tag{2.6}$$
$$\operatorname{Re} W = F_1(\sigma), \ |\sigma| < a, \ \theta = 0$$
$$\operatorname{Re} We^{-i\pi\mu} = 0, \quad |\sigma| > a, \quad \theta = 0$$

As is well-known, solution of this problem yields a regularization of the complete singular equation (third method) /17/. The same solution may be used to find asymptotic solutions of the integral Eq.(2.4).

The above results have several corollaries. Since $U(\sigma,\theta)$ is an even function of σ, we can replace it by its Fourier expansion

$$U(\sigma,\theta)=\int_0^\infty C(k)\,\mathrm{sh}\,k(\theta-\theta_0)\cos k\sigma\,dk$$

inserting it into (1.8). Interchanging the order of integration and using the integral representation for Legendre functions of the first kind /15/, we obtain

$$H(\sigma,\theta)=\mathrm{Re}\,(\mathrm{sh}\,m\sigma)^{2\mu}\frac{\sqrt{\pi}}{2}\Gamma(1/2-\mu)\int_0^\infty C(k)\,\mathrm{sh}\,k(\theta-\theta_0)P_{is}^{\mu}(\mathrm{ch}\,2m\sigma)\,dk \tag{2.7}$$

From (2.7) we conclude that mixed problems for system (0.1) reduce to paired integral equations

$$\mathrm{Re}\int_0^\infty C(k)\,\mathrm{sh}\,k\theta_0P_{is}^{\mu}(\mathrm{ch}\,2m\sigma)\,dk=g(\sigma),\quad \sigma<a \tag{2.8}$$

$$\mathrm{Re}\int_0^\infty C(k)\,\mathrm{ch}\,k\theta_0P_{is}^{\mu}(\mathrm{ch}\,2m\sigma)\,k\,dk=0,\quad \sigma>a,\quad s=k/2m$$

which, as remarked above, can be solved exactly at $2m\theta_0=\pi$.

3. The technique just described to determine an integral representation of solutions of systems

$$\frac{\partial H}{\partial\theta}=-K(\sigma)\frac{\partial\psi}{\partial\sigma},\quad \frac{\partial H}{\partial\sigma}=K(\sigma)\frac{\partial\psi}{\partial\theta}$$

in terms of an arbitrary analytic function is clearly applicable in cases when the solution of the ordinary differential equation

$$H_n''(\sigma)-\frac{K'(\sigma)}{K(\sigma)}H_n'(\sigma)+n^2H_n(\sigma)=0$$

can be represented, after suitable reduction, by an integral

$$H_n(\sigma)=A_1(\xi)\int_0^1\tau^{\alpha}(1-\tau)^{\beta}R(\tau\cdot\xi)\,d\tau,\quad \xi=\xi(\sigma) \tag{3.1}$$

where the constants $\alpha,\beta>-1$ are independent of the parameter n. Special cases of (3.1) are the well-known representations of special functions /15/, with the exception of the Laguerre and Chebyshev-Hermite functions, the hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;t)$ and its generalization ${}_3F_2(\alpha,\beta;c;\gamma;d;t)$.

It follows immediately from (1.18) that the integral operator (1.8) also transforms a solution of the classical heat conduction equation

$$\partial^2U/\partial\sigma^2=a^2\partial U/\partial\theta$$

into a solution of the equation

$$\frac{\partial^2H}{\partial\sigma^2}+\frac{2m(1-2\mu)}{\mathrm{sh}\,2m\sigma}\frac{\partial H}{\partial\sigma}=a^2\frac{\partial H}{\partial\theta}$$

and similarly the operator (2.2) transforms a solution of the Cauchy problem for this equation into a solution of the same problem for the heat conduction equation.

In the hyperbolic case ($A=-1$) the operator (1.8) converts the solution of the wave equation into a solution of system (0.1). Then, applying the appropriate arguments, we see that a solution of the equation

$$\frac{\partial^2H}{\partial\sigma^2}-\frac{\partial^2H}{\partial\theta^2}+\frac{2m(1-2\mu)}{\mathrm{sh}\,2m\sigma}\frac{\partial H}{\partial\sigma}=0$$

can be written in the form

$$H(\sigma,\theta)=(1-\nu)\int_{-1}^{1}(1-u^2)^{-\nu}F(p)\,du+\frac{\operatorname{sh}2m\sigma}{2m}\left\{\int_{-1}^{1}\frac{(1-u^2)^{-\nu-1}f'(p)}{1+u^2\operatorname{sh}^2m\sigma}\,du+\right. \tag{3.2}$$

$$\left.\int_{-1}^{1}\frac{(1-u^2)^{\nu-1}u\operatorname{sh}m\sigma}{(1+u^2\operatorname{sh}^2m\sigma)^{3/2}}f(p)\,du\right\}\left(\frac{\operatorname{th}m\sigma}{m}\right)^{2\nu-2}$$

$$p=\theta+m^{-1}\operatorname{Arsh}u\operatorname{sh}m\sigma,\quad \nu={}^1/_2+\mu$$

where $F(p)$ and $f(p)$ are arbitrary functions. As $m\to 0$ formula (3.2) gives the well-known solution of the Euler-Poisson-Darboux equation /19/. Using this representation, one can easily construct a solution of the generalized Cauchy problem (19/, p. 276) for the equation:

$$H(0,\theta)=(1-\nu)F(\theta)\int_{-1}^{1}(1-u^2)^{-\nu}du$$

$$\lim_{\sigma\to 0}(m\operatorname{cth}m\sigma)^{1-2\mu}\frac{\partial H}{\partial\sigma}(\sigma,\theta)=\left(\nu-\frac{1}{2}\right)f'(\theta)\int_{-1}^{1}(1-u^2)^{\nu-1}du$$

It is quite clear how to extend this result to second-order equations for functions of more than two variables (/19/, p.194). We mention only that a solution of the equation

$$\operatorname{div}[(m\operatorname{cth}m\sigma)^{1-2\mu}\nabla H]=0,\quad H=H(\sigma,\theta,\eta)$$

is transformed by (1.8) into a solution of the three-dimensional Laplace equation which is an even function of σ.

Now consider the following problem for system (0.1) with $\mu=0$:

$$H(\sigma,\theta_0)=0,\quad 0<\delta<\infty$$

$$H(\sigma,0)=P=\text{const},\quad 0<\sigma<a,\quad \frac{\partial H}{\partial\theta}(\sigma,0)=0,\quad \sigma>a$$

and assume that $2m\theta_0=\pi$. This problem may be interpreted as the determination of the inflow to a fissure of length $2a$ in an inhomogeneous layer with permeability $K=m\operatorname{th}m\sigma$ and width $h=2\theta_0$, on the outer boundaries of which the discharge is zero, while the discharge in the fissure, in a high-permeability zone, is a constant P.

By (1.8), (2.2) and (2.6), this boundary-layer problem reduces to the following problem for the harmonic function $U(\sigma,\theta)$:

$$U(\sigma,\theta_0)=0\quad |\sigma|<\infty$$

$$U(\sigma,0)=\begin{cases}2P/\pi & |\sigma|<a\\ 0 & |\sigma|>a\end{cases}$$

Applying Schwarz's formula for poles /18/ and using (1.8), we obtain the solution as

$$H(\sigma,\theta)=\operatorname{Re}\frac{2P}{\pi^2 i}\int_0^{\sigma}\frac{m\operatorname{ch}m\xi}{\sqrt{\operatorname{sh}^2m\sigma-\operatorname{sh}^2m\xi}}\ln\frac{\operatorname{sh}m(\zeta-a)}{\operatorname{sh}m(\zeta+a)}\,d\xi \tag{3.3}$$

$$\zeta=\xi+i\theta$$

and derive a formula for the discharge through the fissure:

$$Q=\psi(a,0)=\frac{P}{\pi m}\ln\frac{1+\operatorname{th}ma}{1-\operatorname{th}ma} \tag{3.4}$$

(The integral is evaluated via the substitution $\operatorname{th}m\xi=\operatorname{th}ma\sin\varphi$.)

As $m\to 0$ $(\theta_0\to\infty)$ it follows from (3.3) and (3.4) that

$$H(\sigma,\theta)=\operatorname{Re}\frac{2P}{\pi^2 i}\int_0^{\sigma}\ln\frac{\zeta-a}{\zeta+a}\frac{d\xi}{\sqrt{\sigma^2-\xi^2}},\quad Q=\frac{2Pa}{\pi} \tag{3.5}$$

corresponding to the classical problem of a circular fissure in a half-space.

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REFERENCES

1. POLOZHII G.N., Theory and Application of p-Analytic and p, q-Analytic Functions, Naukova Dumka, Kiev, 1973.
2. DOMBROVSKII G.A., Method of Approximations of the Adiabatic Curve in the Theory of Two-dimensional Gas Flow, Nauka, Moscow, 1964.
3. BERNADINER M.G. and ENTOV V.M., Hydrodynamic Theory of the Filtration of Anomalous Liquids, Nauka, Moscow, 1975.
4. MIFTAKHUTDINOV B.A., MOLOKOVICH YU.M. and SKVORTSOV E.V., Some questions of two-dimensional stationary non-linear filtration. In: Problems of Hydrodynamics and the Rational Exploitation of Oil Deposits, Izd. Kazan. Univ., Kazan, 1971.
5. ENTOV V.M., PANKOV V.N. and PAN'KO S.V., A contribution to calculation of blocks of residual viscoelastic oil. Prikl. Mat. Mekh., 44, 5, 1980.
6. PANKOV V.N., PAN'KOV S.V. and PODGAINYI V.P., On the construction of exact solutions of boundary-value problems of filtration with limit gradient. In: Boundary-value Problems of Filtration Theory, Abstracts of Lectures, All-Union Conference-Seminar, Uzhgorod, 1976.
7. DOMBROVSKII G.A., On certain systems of first-order equations and the corresponding generalized Euler-Poisson-Darboux equations. Differents. Uravn., 14, 1, 1978.
8. PAN'KO S.V., On exact solutions of boundary-value problems of non-linear filtration. Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gazov. 3, 1985.
9. PAN'KO S.V., On some problems of filtration with limit gradient. Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gazov, 4, 1973.
10. BASAK N.K. and DOMBROVSKII G.A., On solution of filtration problems with limit gradient. Prikl. Mat. Mekh., 47, 6, 1983.
11. UFLYAND A.S., The Method of Paired Integral Equations in Problems of Mathematical Physics, Naüka, Moscow, 1977.
12. ALEKSANDROV V.M. and MKHITARYAN S.M., Contact Problems for Bodies with Thin Coats and Layers, Nauka, Moscow, 1983.
13. POPOV G.YA., Concentration of Elastic Stresses near Stamps, Cuts, Thin inclusions and Reinforcements, Nauka, Moscow, 1982.
14. ALEKSANDROV V.M. and KOVALENKO E.V., Problems in the Mechanics of Continuous Media with Mixed Boundary Conditions, Nauka, Moscow, 1986.
15. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals Sums, Series and Products, Fizmatgiz, Moscow, 1962.
16. ENTOV V.M., On the analogy of the equations of two-dimensional filtration and lengthwise displacement of non-linearly elastic and plastic bodies. Prikl. Mat. Mekh., 34, 1, 1970.
17. GAKHOV F.D., Boundary-value Problems, Nauka, Moscow, 1977.
18. LAVRENT'EV M.A. and SHABAT B.V., Methods of the Theory of Functions of a Complex Variable, Nauka, Moscow, 1973.
18. BITSADZE A.V., Some Classes of Partial Differential Equations, Nauka, Moscow, 1981.

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